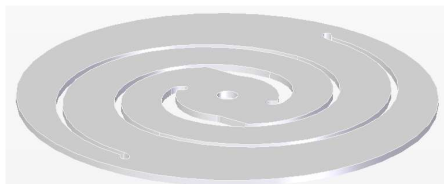
Sage Model Notes

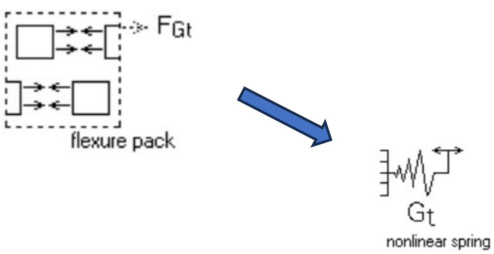
FlexurePack.scfn

D. Gedeon
21 April 2025

A flexure pack is a stack of one or more so-called *flexures*, or planar springs, often with spiral-arm geometries, used to align and resonate moving masses in free-piston stirling-cycle machines. Something like this, for example:



In the Sage edit window the model comprises a submodel with a nonlinear-spring component inside:



The model does not actually design flexures. Rather it allows you to scale the dimensions of an existing flexure to meet different design constraints. Designing flexures requires FEA analysis in a CAD environment. But it is not a trivial task because the effects of shape and dimensional changes on spring stiffness, peak stress and natural frequency are tightly interrelated, making it difficult to accomplish design goals by just guessing. This model can eliminate much of the guesswork by implementing a few relatively simple scaling rules that are reasonably accurate for any type of flexure, regardless of spiral-arm shape, number, winding angle, etc.

The basic idea is to start with a fully designed protype flexure defined by a set of parameters:

D	Outer diameter
t	Thickness
K	Static spring stiffness
F_n	1st mode natural frequency
x_m	Max amplitude (at stress limit)

Then, *assuming the geometrical shape profile remains the same*, define outer diameter and natural frequency as independent inputs and scale the remaining parameters as dependent outputs, according to relationships for stored energy, spring stiffness and peak stress.

The *flexure pack* submodel defines several user-defined inputs and outputs:

```
Inputs
  Ddesign          design diameter (m)                        5.000E-02
  FnDesign         design natural frequency (Hz)              2.000E+02
  Nsprings         number of springs (NonDim)                 1.000E+00
  Dref             reference diameter (m)                     1.000E-01
  ThkRef           reference thickness (m)                    1.300E-03
  Kref             reference stiffness (N/m)                  7.900E+03
  FnRef            reference natural frequency (Hz)           1.000E+02
  XmRef            reference max deflection (m)               6.300E-03
```

The last 5 inputs are reference parameters pertaining to a baseline flexure of your choice, vetted by FEA and possibly actual testing. The above values will likely not be meaningful for your chosen baseline flexure shape. But once properly set they form the basis for an infinite number of flexure designs, characterized by the scaled output values:

```
Outputs
  XmScaled         scaled max deflection                      2.222E-03
    XmRef * (FnRef / FnDesign)
  ThkScaled        scaled thickness                           3.307E-04
    ThkRef * Sqr(Ddesign / Dref) * (XmRef / XmScaled)
  Kscaled          scaled static stiffness                    2.086E+03
    Kref *  (ThkScaled / ThkRef)*Sqr(ThkScaled / ThkRef) *
Sqr(Dref/Ddesign)
  Kpack            effective pack spring stiffness            4.633E+04
    Nsprings * Kscaled * (1 - Sqr(Freq/FnDesign )  )
```

The final output `Kpack` is the effective stiffness of the flexure pack after subtracting that part of the stiffness required to resonate its own effective reciprocating mass. The F_{G_t} force connector emerging from the flexure pack will typically be connected to a reciprocating mass component in a larger Sage model. The mass of that component need only include every physical object moving along with it, except for the flexures themselves. This is true to the extent the modal-vibration shape of the baseline flexure in the FEA modal analysis by which you determined its natural frequency is close to the operating deflection shape. At the very least these requires that the flexure is clamped the same way at the center and outer rim. More on that below.

The inputs of the *nonlinear spring* within the submodel are overridden in terms of the submodel outputs:

```
Recasts
  K0 = Kpack
  Xm = XmRef
```

To use this sample model you can copy and paste it into a larger model of a free-piston machine, then adjust inputs `Ddesign, FnDesign, Nsprings` by hand or using the Sage optimizer to resonate the reciprocating mass it is connected to at the required frequency, subject to whatever constraints your application imposes.

Scaling Relationships

The equations governing the scaling of x_m, t and K from D and F_n originate in the flexures section of the Sage User's Guide and an energy conservation principle (below), which concludes that the ratio of peak stress τ_m to natural frequency ω_n in a flexure spring is proportional to peak displacement x_m, according to the simple relationship

$$\frac{\tau_m}{\omega_n} \propto x_m\sqrt{\rho E} \tag{1}$$

where ρ is material density and E is elastic modulus. This conclusion is universally true for any type of spring, as long as the stresses are similarly distributed within the spring. Presuming peak stress and material properties are maintained constant in a scaled spring design, this equation amounts to a simple relationship for scaling max deflection:

$$x_m \propto \frac{1}{\omega_n} \tag{2}$$

Two additional relationships in the Sage User's Guide for peak stress and axial stiffness are the basis for scaling the flexure stiffness and spring rate

$$t \propto \frac{D^2}{x_m} \tag{3}$$

$$K \propto \frac{t^3}{D^2} \tag{4}$$

The formulas in the Sage manual are a bit more complicated but under the assumption of geometrically-similar scaling, some parameters are constant or cancel in pairs because they scale by the same ratios.

Energy Conservation Derivation

The stored energy per unit volume in a material subject to stress τ and strain ϵ is

$$\mathrm{e} \propto \int_0^{\epsilon_\mathrm{m}} \tau \mathrm{d}\epsilon \propto \int_0^{\tau_\mathrm{m}} \tau\frac{\mathrm{d}\tau}{\mathrm{E}} = \frac{1}{2\mathrm{E}}\tau_\mathrm{m}^2$$

Where τ_m is the peak stress and E is the elastic modulus. The integration limit ϵ_m is the maximum strain. This is relatively obvious if stress is a unidirectional tensile stress (as in a stretched rod) but it also applies to shear stresses if elastic modulus E is replaced with shear modulus G. This is not a fundamental difference because E and G are proportional for a given material.

Multiplying by the effective volume $V = M/\rho$ of the material under stress (or actual volume if the stress is uniformly distributed) gives the total stored energy.

$$\mathrm{E} \propto \frac{\mathrm{V}}{2\mathrm{E}}\tau_\mathrm{m}^2 = \frac{\mathrm{M}}{2\rho\mathrm{E}}\tau_\mathrm{m}^2$$

On the other hand, **the stored energy in a spring** with stiffness K and peak deflection x_m is given by the well-known formula

$$E = \frac{1}{2}Kx_m^2$$

Equating the two stored energies gives the relationship

$$\frac{M}{\rho E}\tau_m^2 \propto K\,x_m^2$$

Or regrouping,

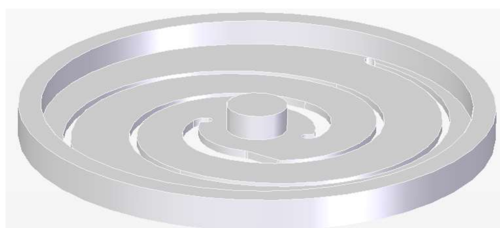
$$\frac{K}{M} \propto \frac{\tau_m^2}{\rho E\,x_m^2}$$

But K/M is just the square of the natural frequency ω_n^2. Making this substitution and taking the square root gives equation (1) above.

One take-away from this analysis is that if you are having trouble designing a flexure to survive both high amplitude and high frequency operation you should design the arms to distribute stress as uniformly as possible (similar to a coil spring) or reduce the ρE product by using a material of lower density and elastic modulus (e.g. peek, compared to steel).

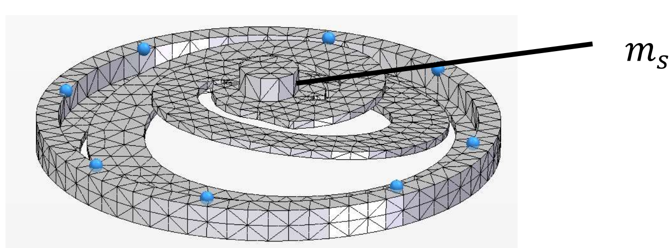
Flexure FEA

CAD software typically includes linear-static and normal-mode analysis tools to calculate bending shapes resulting from applied forces and frequencies of resonant vibration modes. In the case of flexures it is important that the analysis reflect the constraints applied to the installed flexure — namely the clamping by washers or other structures at the outer rim and center. The CAD rendering below shows a flexure with stiffening elements added at the rim and center that represent the clamped regions.



These elements affect both the static spring stiffness and modal vibration frequencies.

The goal of FEA modal analysis is to characterize the flexure spring stiffness and effective vibrating mass for design purposes. The spring stiffness K is immediately available from linear-static FEA of the stiffened flexure model. The effective vibrating mass requires subtracting the added mass m_s of the stiffening element from the vibrating mass inferred from the modal-analysis resonant frequency ω_s for the stiffened structure.



In math-speak, the total effective reciprocating mass is $m_s + m_f$, where m_f is the effective reciprocating mass of the flexure without the stiffening element. From the general resonant frequency relation ship $K = m\omega^2 = (m_f + m_s)\omega_s^2$ it follows that

$$m_f = \frac{K}{\omega_s^2} - m_s$$

The resonant frequency of the flexure that bends the same way without the stiffening element is

$$\omega_f = \sqrt{\frac{K}{m_f}} \tag{5}$$

Values m_f and ω_f are useful for design purposes.

To be especially careful you should run modal analyses with different values for the stiffening element mass m_s to make sure that ω_f does not depend on m_s. If it does, it means that the bending shape of the arms themselves depends on the resonant frequency. In that case you should adjust m_s so the modal-analysis frequency matches the target operating frequency.

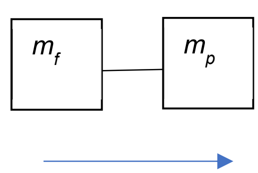
Effective Spring Stiffness

Output `Kpack` of the Sage model calculates an effective spring stiffness K_e in terms of the static stiffness K, the actual operating frequency ω and the above flexure natural frequency ω_f

$$K_e = K\left(1-\left(\omega/\omega_f\right)^2\right) \tag{6}$$

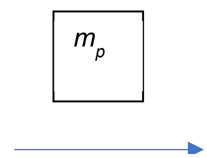
Stiffness K_e is what is available to accelerate additional *payload* reciprocating mass attached to the flexure after subtracting what is required to accelerate its own effective mass m_f. This is at-least plausible because the effective stiffness approaches zero as the operating frequency approaches ω_f, where the flexure has only enough stiffness to accelerate itself.

To make this clear the following sketch represents a spring with static stiffness K sinusoidally accelerating a reciprocating mass m_p with flexure effective mass m_f included separately



$$K = \left(m_f + m_p\right)\omega^2$$

The next sketch represents the case where the effective mass is not included and an effective spring K_e accelerates only the reciprocating mass m_p, to achieve the same acceleration



$$\mathrm{K_e} = m_p \omega^2$$

Solving the equation in the first sketch for $m_p\omega^2$ and substituting in the equation in the second sketch gives

$$K_e = \mathrm{K} - m_f \omega^2$$

Then substituting equation (5) for m_f gives equation (6).